Theoretical progress on $\pi\pi$ scattering lengths and phases

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Outline

Why is $\pi\pi$ scattering interesting

- the pions are the quasi-Goldstone bosons of spontaneous chiral symmetry breaking of QCD
- their interaction vanishes in the limit of zero momenta and quark masses
- a precision study of the departure from this limit thoroughly tests our understanding of strong interactions in the nonperturbative regime (*e.g.* through lattice calculations)
- multipion final states are ubiquitous in hadronic decays: understanding the $\pi\pi$ interaction is important for many other reactions (*e.g.* $K \to 2\pi, 3\pi$, $\eta \to 3\pi$, etc.)
- at low energy the two S-wave scattering lengths are the essential parameters: *e.g.* the parameters of the σ resonance are determined, in a model-independent way, by a_0^0 and a_0^2 Caprini, GC, Leutwyler (06)

Determination of the σ resonance parameters

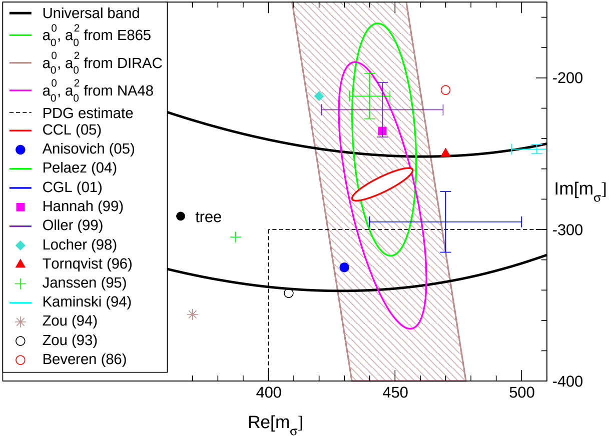

Figure from H. Leutwyler

Low-energy theorem for $\pi\pi$ scattering

$$\mathcal{M}(\pi^0\pi^0 \to \pi^+\pi^-) \equiv A(s,t,u) = \text{isospin invariant amplitude}$$

Low energy theorem: $$A(s,t,u) = \frac{s - M^2}{F^2} + \mathcal{O}(p^4)$$ Weinberg 1966

$$M^2 = B(m_u + m_d) \qquad M_\pi^2 = M^2 + O(m_q^2), \;\; F_\pi = F + O(m_q)$$

All physical amplitudes can be expressed in terms of $A(s,t,u)$

$$T^{I=0} = 3A(s,t,u) + A(t,s,u) + A(u,t,s) \;\Rightarrow\; T^{I=0} = \frac{2s - M_\pi^2}{F_\pi^2}$$

S wave projection (I=0)

$$t_0^0(s) = \frac{2s - M_\pi^2}{32\pi F_\pi^2} \qquad a_0^0 = t_0^0(4M_\pi^2) = \frac{7M_\pi^2}{32\pi F_\pi^2} = 0.16$$

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All physical amplitudes can be expressed in terms of $A(s,t,u)$

$$T^{I=2} = A(t,s,u) + A(u,t,s) \;\Rightarrow\; T^{I=2} = \frac{-s + 2M_\pi^2}{F_\pi^2}$$

S wave projection (I=2)

$$t_0^2(s) = \frac{2M_\pi^2 - s}{32\pi F_\pi^2} \qquad a_0^2 = t_0^2(4M_\pi^2) = \frac{-M_\pi^2}{16\pi F_\pi^2} = -0.045$$

Chiral predictions for a_0^0 and a_0^2

Quark mass dependence of M_π and F_π:

$$
\begin{aligned}
M_\pi^2 &= M^2 \left(1 - \frac{M^2}{32\pi^2 F^2}\bar{\ell}_3 + O(M^4)\right) \\
F_\pi &= F \left(1 + \frac{M^2}{16\pi^2 F^2}\bar{\ell}_4 + O(M^4)\right)
\end{aligned}
$$

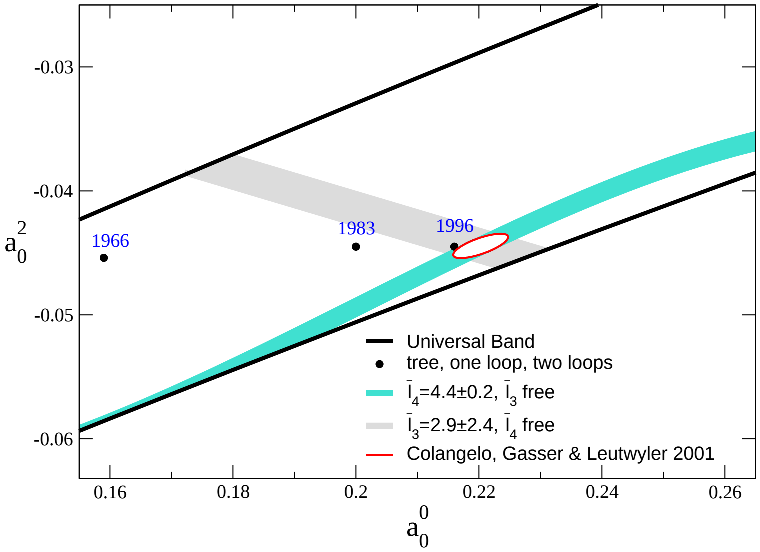
Phenomenological determinations (indirect):

$$\bar{\ell}_3 = 2.9 \pm 2.4$$ Gasser & Leutwyler (84)

$$\bar{\ell}_4 = 4.4 \pm 0.2$$ GC, Gasser & Leutwyler (01)

Lattice calculations determine these constants directly

Chiral predictions for a_0^0 and a_0^2



Sensitivity to the quark condensate

The constant $\bar{\ell}_3$ determines the NLO quark mass dependence of the pion mass

$$M_\pi^2 = 2B\hat{m}\left[1 + \frac{2B\hat{m}}{16\pi F_\pi^2}\bar{\ell}_3 + \mathcal{O}(\hat{m}^2)\right]$$

$$\hat{m} = \frac{m_u + m_d}{2} \qquad B = -\frac{1}{F^2}\langle 0|\bar{q}q|0\rangle$$

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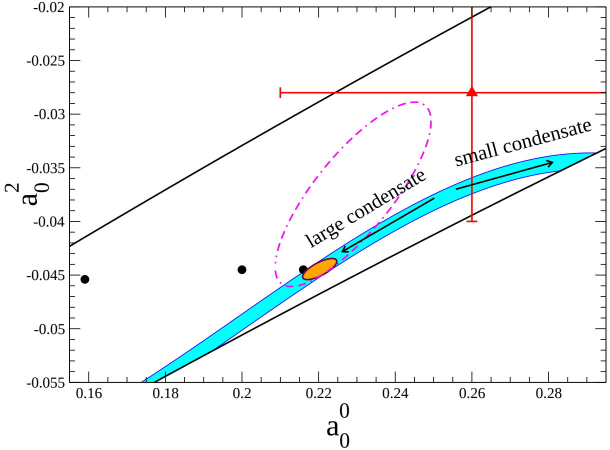
Its size tells us what fraction of the pion mass is given by the Gell-Mann–Oakes–Renner term

$$M_{\rm GMOR}^2 \equiv 2B\hat{m}$$

or how large is the quark condensate, the order parameter of chiral symmetry breaking.

Jan Stern and collaborators have emphasized this since long!

Sensitivity to the quark condensate



The E865 data on $K_{\ell 4}$ imply that

GC, Gasser and Leutwyler PRL (01)

$$M_{\mathrm{GMOR}} > 94\% M_\pi$$

Situation after new data?

Higher orders

Higher order corrections are suppressed by $\mathcal{O}(m_q^2/\Lambda^2)$
$\Lambda \sim 1$ GeV $\Rightarrow$ expected to be a few percent

$$a_0^0 = 0.200 + \mathcal{O}(p^6) \qquad a_0^2 = -0.0445 + \mathcal{O}(p^6)$$

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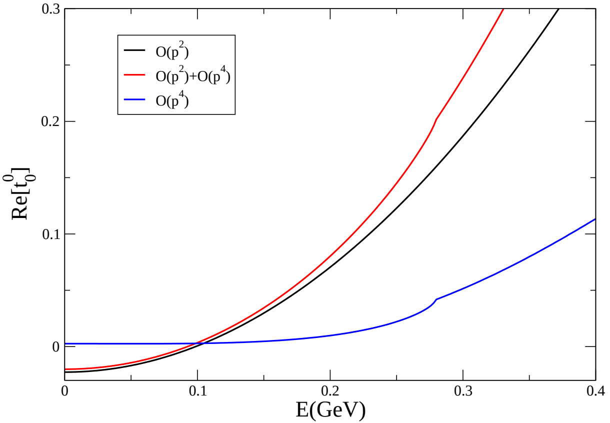
The reason for the rather large correction in a_0^0 is a chiral log

$$a_0^0 = \frac{7M_\pi^2}{32\pi F_\pi^2}\left[1 + \frac{9}{2}\ell_\chi + \ldots\right] \qquad a_0^2 = -\frac{M_\pi^2}{16\pi F_\pi^2}\left[1 - \frac{3}{2}\ell_\chi + \ldots\right]$$

$$\ell_\chi = \frac{M_\pi^2}{16\pi^2 F_\pi^2}\ln\frac{\mu^2}{M_\pi^2}$$

Gasser and Leutwyler (84)

Higher orders



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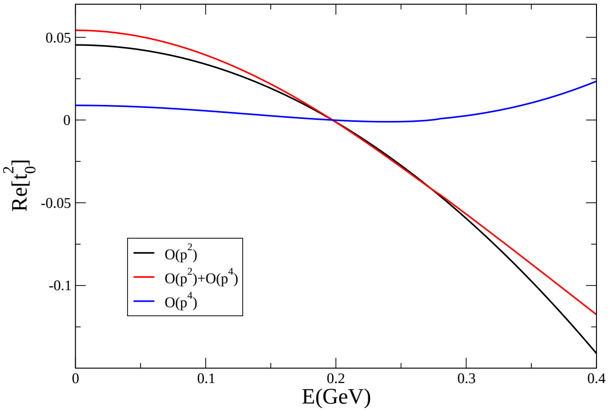

Roy equations

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two subtraction constants, *e.g.* a_0^0 and a_0^2

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Note: if a_0^0, a_0^2 are chosen within the universal band
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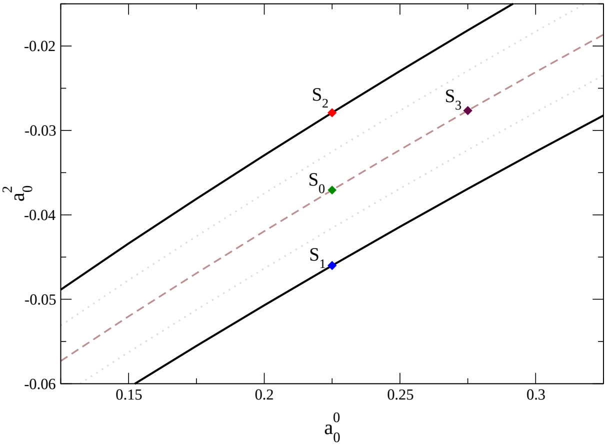
Numerical solutions of the Roy equations
Pennington-Protopopescu, Basdevant-Froggatt-Petersen (70s)
Ananthanarayan, GC, Gasser and Leutwyler (00)
Descotes-Genon, Fuchs, Girlanda and Stern (01)

Numerical solutions



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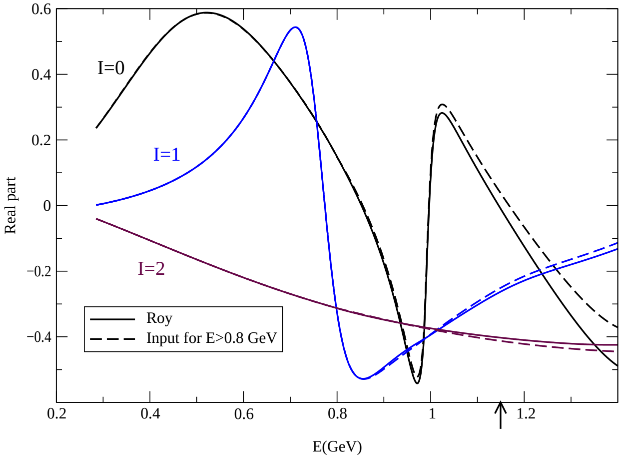

Combining CHPT and dispersive methods

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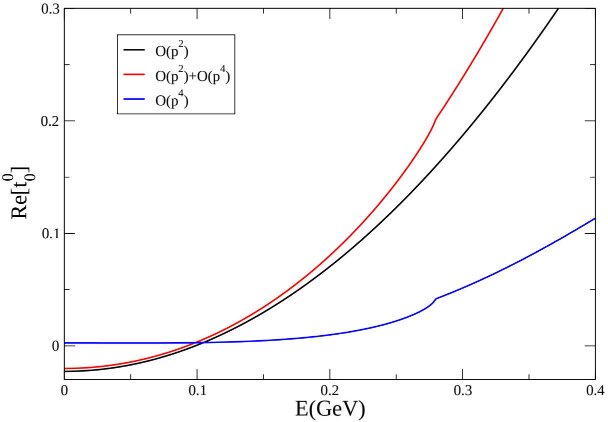
The freedom in the choice of the subtraction point
can be exploited to use the chiral expansion
where it converges best, *i.e.* below threshold

Combining CHPT and dispersive methods



Combining CHPT and dispersive methods

The convergence of the series at threshold is greatly improved if CHPT is used only below threshold

CHPT at threshold

$$\begin{aligned} a_0^0 &= \quad 0.159 \to \quad 0.200 \to \quad 0.216 \\ 10 \cdot a_0^2 &= -0.454 \to -0.445 \to -0.445 \\ & \qquad p^2 \qquad\qquad p^4 \qquad\qquad p^6 \end{aligned}$$

GC, Gasser and Leutwyler (01)

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CHPT below threshold + Roy

$$\begin{aligned} a_0^0 &= \quad 0.197 \to \quad 0.2195 \to \quad 0.220 \\ 10 \cdot a_0^2 &= -0.402 \to -0.446 \to -0.444 \end{aligned}$$

GC, Gasser and Leutwyler (01)

Final results

$$\begin{aligned} a_0^0 &= 0.220 \pm 0.001 + 0.027\Delta_{r^2} - 0.0017\Delta\ell_3 \\ 10 \cdot a_0^2 &= -0.444 \pm 0.003 - 0.04\Delta_{r^2} - 0.004\Delta\ell_3 \end{aligned}$$

where

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Adding errors in quadrature $[\Delta_{r^2} = 6.5\%,\ \Delta\ell_3 = 2.4]$

$$a_0^0 = 0.220 \pm 0.005$$
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Pelaez and Yndurain have criticized these results

Claim 1: our input above 1.4 GeV is not correct (PY 03)

The criticism has been answered (Caprini *et al.* 03)

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Claim 2: our calculation for $\langle r^2\rangle_s$ is not correct (Y, 04)

The criticism has been answered (Ananthanarayan *et al.* 04)

The analysis of Stern *et al.*

- Stern and collaborators advocate that it is even more interesting not to attempt any (indirect) determinations of $\bar{\ell}_3$ and $\bar{\ell}_4$
- they also use the solutions of the Roy equations in order to analyze the data, and with them translate low-energy data into values of the scattering lengths
- our two independent numerical solutions of the Roy equations agree – the outcome of our analyses agree also

The analysis of Peláez and Ynduráin

Peláez and Ynduráin have proposed a different approach and analyze the data with a parametrization which

- is simple, fits the data and has the correct cut structure at low $s > 0$
- approximately satisfies forward dispersion relations
- does not take into account chiral symmetry constraints

Disregarding technical differences, a few essential remarks:

- data at various energies are treated democratically – on the other hand some sets of data are clearly inconsistent with each other
- no use of crossing symmetry – the left-hand cut is not properly implemented
- the use of dispersion relations is limited – it is not required that they are satisfied exactly – in a sense, data and theory are also treated democratically

Other analyses

- Kamiński, Leśniak and Loiseau have also worked out numerical solution of the Roy equations with the aim of resolving an ambiguity among possible phase-shift solutions in the analysis of $\pi N \to \pi\pi N$ data (Cracow-Cern-Munich)
- various other parametrizations/analyses of the $\pi\pi$ scattering amplitude exist in the literature, constructed with different goals
  e.g. D. Bugg (96,05,06), Maiorov and Patarakin (03,05), Achasov and Kiselev (05), etc.

Numerical comparison

Phenomenological analyses

	DFGS	KLL	PY
a_0^0	0.228 ± 0.032	0.224 ± 0.013	0.230 ± 0.015
$-10 \cdot a_0^2$	0.382 ± 0.038	0.343 ± 0.036	0.480 ± 0.046
$(\delta_0^0 - \delta_0^2)\vert_{s=M_K^2}$	47.1°	$37^\circ - \delta_0^2(M_K^2)$ $< 49^\circ$	$52.9^\circ \pm 1.6^\circ$

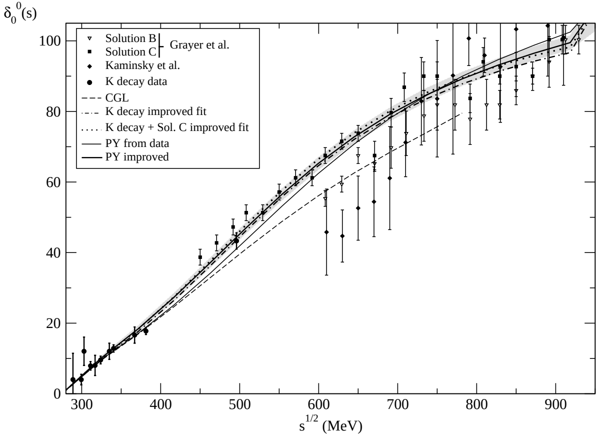
Analysis based on chiral symmetry

	CGL
a_0^0	0.220 ± 0.005
$-10 \cdot a_0^2$	0.444 ± 0.010
$(\delta_0^0 - \delta_0^2)\vert_{s=M_K^2}$	$47.7^\circ \pm 1.5^\circ$

DFGS=Descotes-Genon, Fuchs, Girlanda and Stern, KLL=Kamiński, Leśniak and Loiseau,

PY=Peláez and Ynduráin

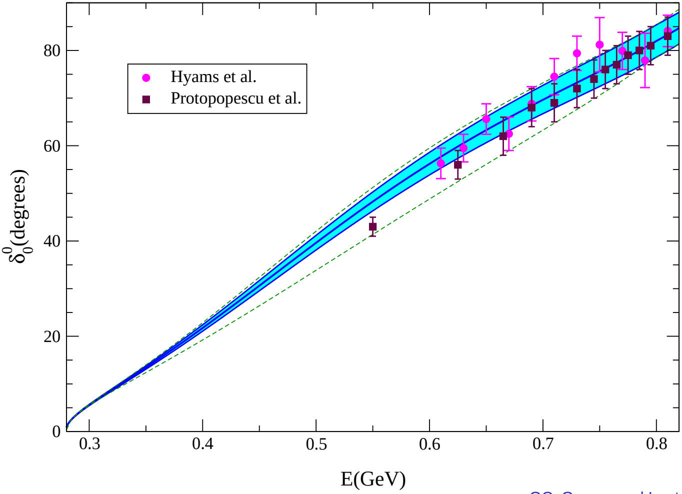
Phase shifts



Peláez and Ynduráin (04)

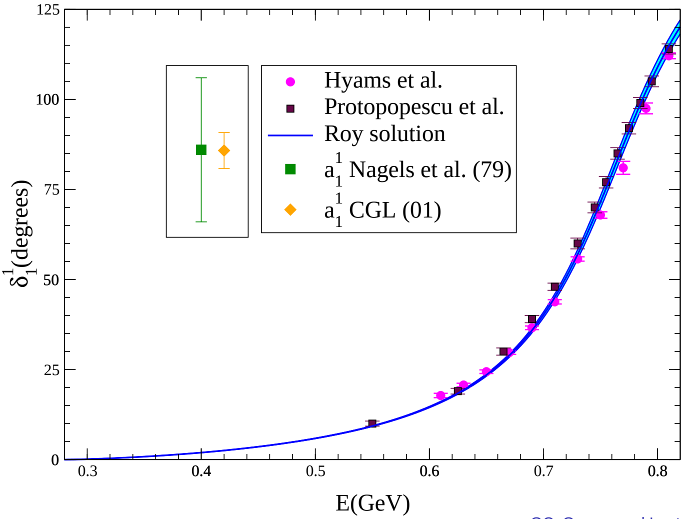
The "shoulder" is incompatible with dispersion relations Leutwyler (06)

Phase shifts



GC, Gasser and Leutwyler (01)

Phase shifts



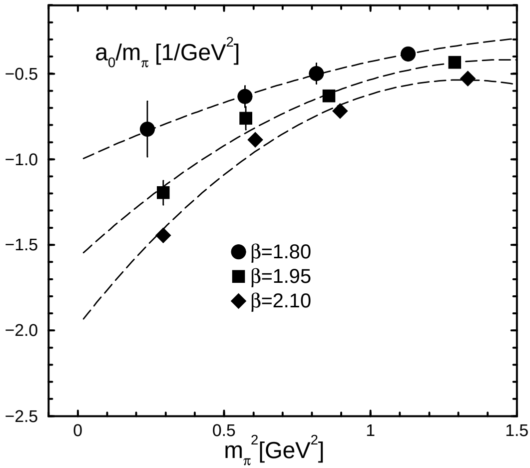
GC, Gasser and Leutwyler (01)

The *P*-wave phase is relevant for $a_\mu^{\rm hvp}$

Lattice calculations of the $\pi\pi$ scattering lengths

- CP-PACS (04):
  - lattice calculation with $N_f = 2$, $O(a)$ improved dynamical quarks
  - continuum and chiral extrapolation performed numerically
  - smallest pion mass: $M_\pi = 540$ MeV
  - calculation of phase shifts also performed
- NPLQCD (05):
  - lattice calculation over configurations of $N_f = 3$, staggered dynamical quarks
  - valence quarks are domain wall fermions
  - no continuum extrapolation (only one lattice spacing) chiral extrapolation performed numerically
  - smallest pion mass: $M_\pi = 294$ MeV

CP-PACS calculation



a_0 here stands for a_0^2

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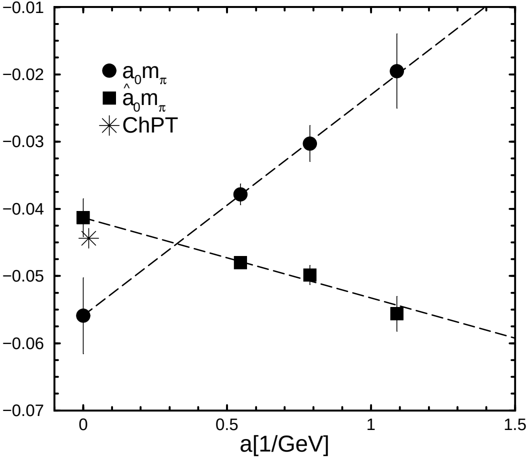

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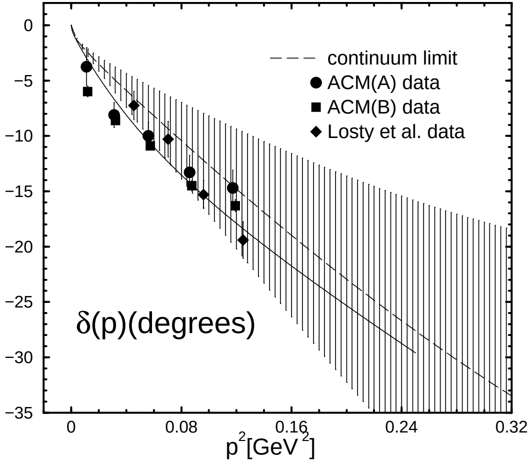
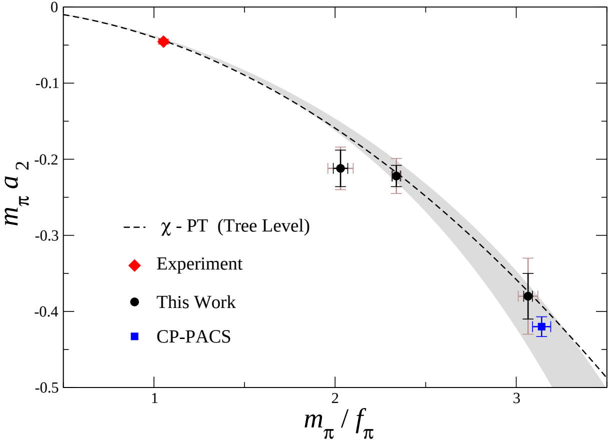


δ here stands for δ_0^2

NPLQCD calculation



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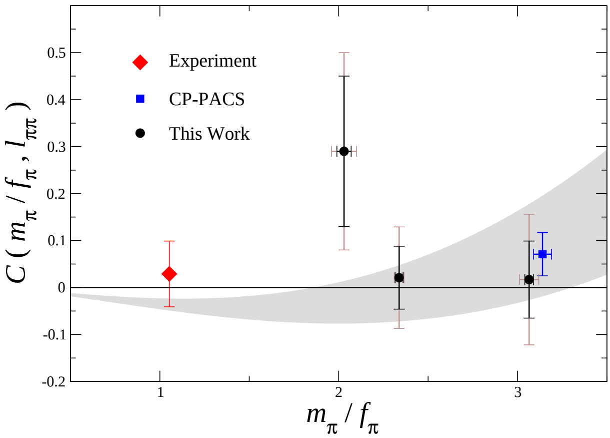


$C \equiv a_0^2/a_0^2(\mathrm{LO}) - 1$

Lattice calculations of $\bar{\ell}_3$ and $\bar{\ell}_4$

- MILC
  - $N_f = 3$ staggered fermions [fourth root trick] determination of the L_i's ($SU(3)$ constants)
  - continuum and chiral extrapolation done numerically and with the help of CHPT – finite volume corrected
  - smallest pion mass: $M_\pi = 240$ MeV
- Lüscher *et al.*
  - $N_f = 2$ Wilson fermions
  - continuum and chiral extrapolation done numerically and with the help of CHPT – finite volume corrected
  - smallest pion mass: $M_\pi = 380$ MeV
- ETM collaboration
  - $N_f = 2$ twisted mass fermions
  - no continuum extrapolation, chiral extrapolation done numerically and with the help of CHPT – finite volume corrected
  - smallest pion mass: $M_\pi \sim 300$ MeV

Lattice calculations of $\bar{\ell}_3$ and $\bar{\ell}_4$

- MILC

$$\bar{\ell}_3 = 0.6 \pm 1.2 \; , \quad \bar{\ell}_4 = 3.9 \pm 0.5$$

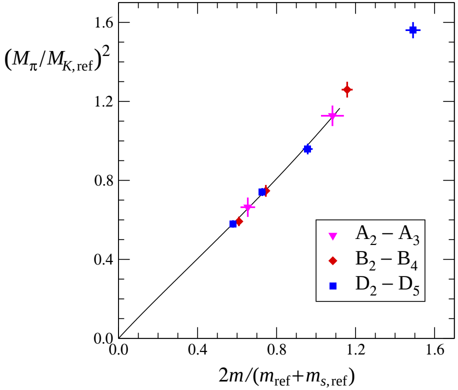
- Lüscher *et al.*

$$\bar{\ell}_3 = 3.5 \pm 0.5 \pm 0.1$$

- ETM collaboration

$$\bar{\ell}_3 = 3.65 \pm 0.12 \; , \quad \bar{\ell}_4 = 4.52 \pm 0.06$$

Lüscher *et al.* calculation



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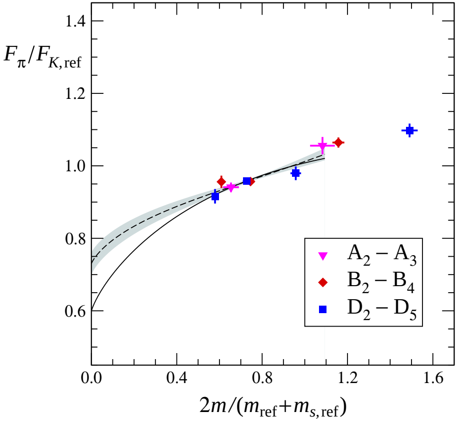
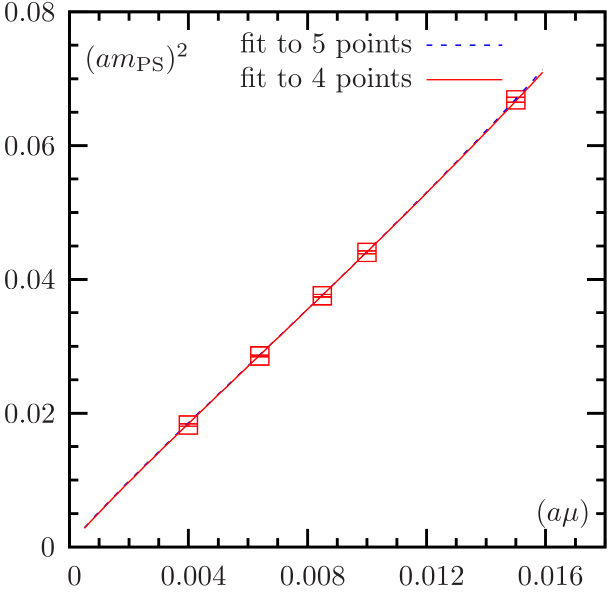


ETM calculation



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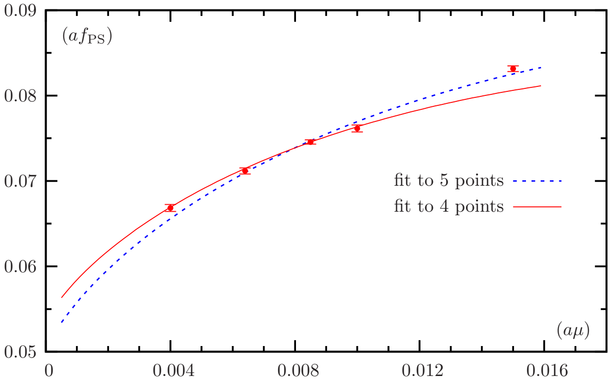
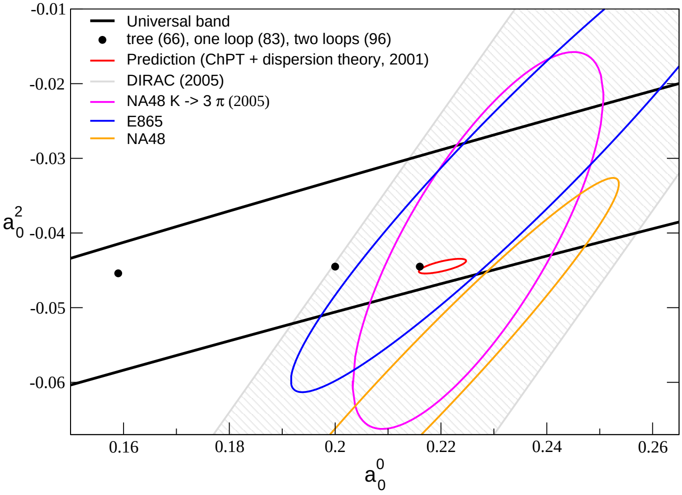


Summary: theory vs experiment



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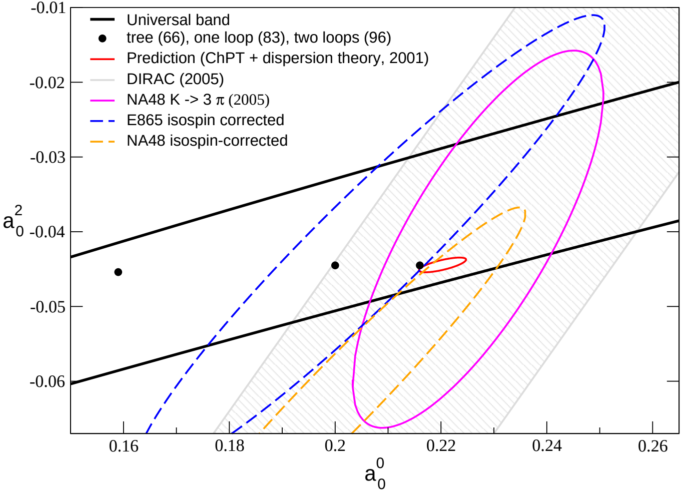
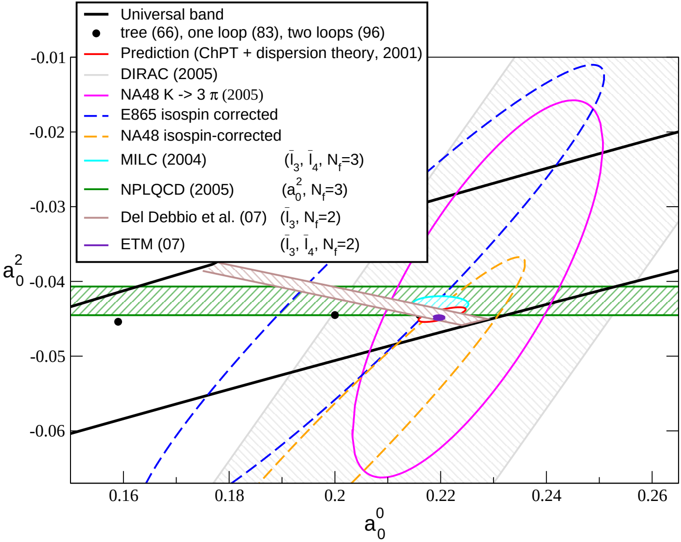


cf. J. Gasser's talk

Summary: lattice vs theory vs experiment



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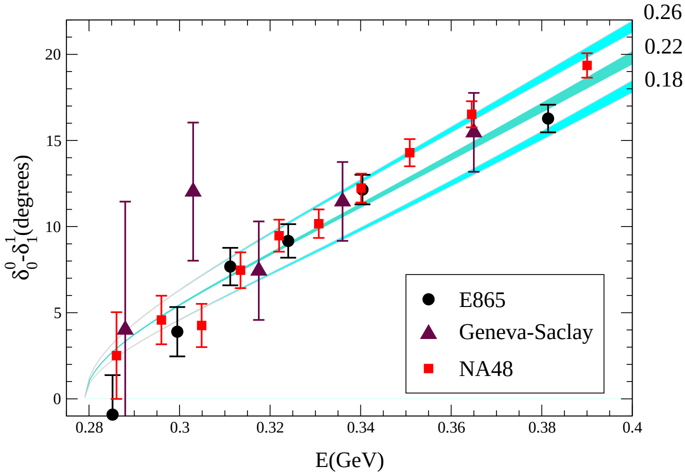
- The prediction relies on the assumption that the Gell-Mann–Oakes–Renner term dominates the pion mass
- Experimental data are approaching the same level of precision and thereby test the underlying assumptions about the structure of the QCD vacuum
- Today even the direct comparison to first principle QCD calculations is possible. I have reviewed recent lattice calculations of the $I = 2$ scattering length and of the quark mass dependence of F_π and M_π

Comparison to the K_{e4} data



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