QCD at low energy: the simplicity of complex non-perturbative phenomena

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Abstract

These lectures provide a short introduction to chiral perturbation theory and to the effective field theory method in general. The focus is mainly on the foundations of the method and only at the end I discuss two recent applications. In a school about complexity, effective field theories provide a nice example of how one can describe complicated nonperturbative phenomena in simple terms.

1 Introduction

Effective field theories are an ubiquitous tool in physics. Indeed very rarely does one hope to describe a physical system at all length scales (or energies), and if one concentrates on a certain range and considers only the modes which get excited at those energies, then one is doing an effective theory. Although the atom contains quarks and gluons (inside the nucleus, that is), we do not need to be able to describe the strong interactions if we want to understand atomic spectra or how several atoms interact and form bound states.

In particle physics we do have a theory that could in principle be valid at all length scales, QCD. In the real world quarks and gluons do have also other types of interactions and interact with other forms of matter too, but one could imagine a world which is completely governed by QCD and where

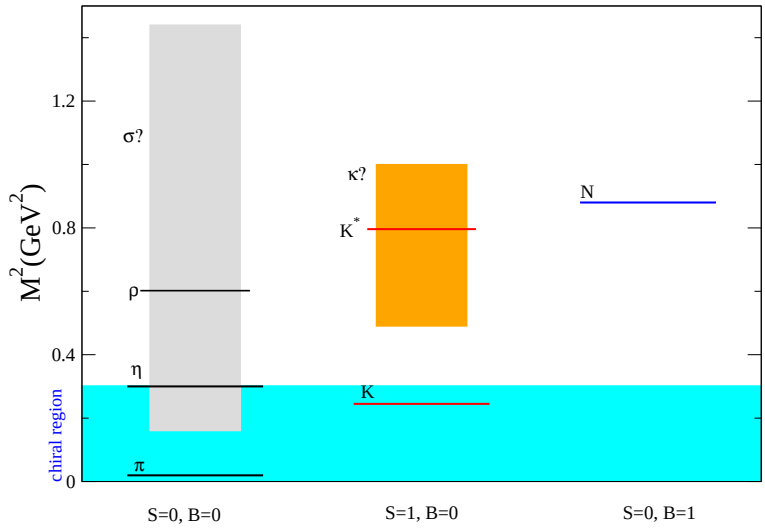

Figure 1: The lowest-lying part of the QCD spectrum in the sectors of strangeness and baryon number 0 and 1. The bands for the σ and the κ indicate the uncertainty in the mass as given by the PDG.

nothing else happens. This imaginary world is realized by the people who do lattice QCD calculations. Of course they do not do this to see what happens at very small length scales, but rather to test how well this theory describes the non-perturbative phenomena which occur at large length scales. At this school Richard Kenway [1] has explained how lattice calculations are performed and what the most recent developments in the field are. He showed that in recent years there have been a number of important developments which make a direct comparison to the phenomenology finally possible. As always, however, the progress is still rather slow, despite the steady increase in computer power, improvement in algorithms and clever new ideas by physicists. The difficulty of this enterprise lies in the fact that one wants to see new degrees of freedom emerge at low energy out of the local interactions among quarks and gluons. We lack analytical tools to do that and the numerical route is understandably not easy.

At very low-energy, however, the complex nonperturbative phenomena which generate hadrons out of quarks and gluons and rule their interactions, become simple, again. This happens because QCD undergoes a spontaneous symmetry breaking: although in the chiral limit the Lagrangian is invariant

under $SU(N_f)_L \times SU(N_f)_R$ (with N_f the number of light flavours) transformations, the vacuum is invariant only under the vector part of this large group. As predicted by the Goldstone theorem massless particles appear in the spectrum, and their interaction becomes weak at low energy. There are no massless hadrons in nature, but an octet of light ones (as illustrated in Fig. 1), which have exactly the quantum numbers predicted by the Goldstone theorem. Their small mass is readily understood in terms of a small mass term in the QCD Lagrangian – this explicit chiral symmetry breaking generates a small mass for the Goldstone bosons. Since the interactions among these Goldstone bosons vanish in the limit of zero quark masses and zero momenta, one can do a perturbation theory around this limit and expand in small masses and momenta. This is the main idea of chiral perturbation theory (CHPT) which I will illustrate in some more details in what follows. As shown in Fig. 1 the theory can be applied in the energy region where only the Goldstone bosons are excited.

Since the method is completely general and can be applied to any system which undergoes spontaneous symmetry breaking, I will put an emphasis on this and describe its foundations without making specific reference to QCD as long as this is possible (even if for convenience I may sometimes call the Goldstone bosons "pions"). The plan of the paper is as follows. In Sect. 2 I will discuss the Goldstone theorem and how one can construct an effective field theory on its basis. In Sect. 3 I will specifically concentrate on the quantum character of the effective field theory and discuss in detail the role and meaning of loop diagrams and how one can carry out the renormalization program. The latter question is particularly relevant in view of the fact that this kind of effective field theories is intrinsically non-renormalizable. In Sect. 4 I will discuss a few recent results in CHPT and finally give an outlook and draw a few conclusions in Sect. 5.

2 Effective field theories for systems with spontaneous symmetry breaking

2.1 Goldstone theorem

The Goldstone theorem is one of the few physics results which are absolutely general and most relevant in very many different fields. If the Hamiltonian of a system is invariant under a group G of global transformations, while the

vacuum only under a subgroup H, then the spectrum of the system contains massless excitations. The number of these depends only on the rank n_G of the group G and that of the subgroup H, n_H, and is given by the difference of the two: $n_{\rm GB} = n_G - n_H$. These massless excitations are usually referred to as Goldstone bosons (which explains the subscript GB above). We indicate with Q_i, $i = 1, \ldots, n_G$ the generators of the group G, with H_i, $i = 1, \ldots n_H$ those of H, and with X_i, $i = 1, \ldots, n_{\rm GB}$, the broken ones – the states obtained by letting these act on the vacuum do not give back the vacuum:

$$X_i|0\rangle \neq |0\rangle \ . \tag{1}$$

On the other hand, since G is a symmetry of the Hamiltonian $\mathcal{H}$, the latter commutes with all the generators, which implies that all the states $Q_i|0\rangle$ are degenerate with the vacuum:

$$\mathcal{H}X_i|0\rangle = X_i\mathcal{H}|0\rangle = 0 \ . \tag{2}$$

The states $X_i|0\rangle$ are the Goldstone bosons. The generators X_i do not form an algebra, and therefore there is no subgroup of G which is generated by them – they generate the coset space G/H, and there is a one–to–one mapping between the elements of this space and the Goldstone boson fields. The existence of this mapping dictates the transformation properties of the Goldstone boson fields under the group G [2]. Without entering the details, we just stress that these are nonlinear.

There is a second part to the Goldstone theorem, which is equally important as the first, best known one. If we denote the one-particle Goldstone boson states by $|\pi^a\rangle$ and consider the matrix elements of the Noether currents J_i^μ of the symmetry group G between the vacuum and these states:

$$\langle 0|J_i^\mu(0)|\pi^a(p)\rangle = iF_i^a p^\mu \tag{3}$$

these define a $n_{\rm GB} \times n_G$ matrix F_i^a. The Goldstone theorem says that this matrix is of maximal rank, *i.e.* of rank $n_{\rm GB}$. In other words, in the proper basis, the Noether currents of the broken generators have nonzero matrix elements with the corresponding pion fields: $F_b^a = \delta_{ab}F_a$. Together with the fact that the Noether currents are conserved, this seemingly uninteresting statement implies that the Goldstone bosons interact weakly at low energy: more precisely, the strength of their interaction vanishes with the square of the momenta.

The construction of the effective Lagrangian for systems with spontaneous symmetry breaking is based on the Goldstone theorem. The first part implies that there are massless particles in the spectrum – if these are the only ones, then we can construct an effective Lagrangian containing only these degrees of freedom. Moreover, the symmetry transformation properties of the Goldstone bosons imply that the Lagrangian has to be nonlinear in the corresponding fields – therefore, it is necessarily nonrenormalizable. This would seem to destroy the usefulness of such an approach, because nonrenormalizability is often connected to lack of predictivity. Here is where the second part of the theorem plays an important role: since the interaction vanishes with the square of the momenta, in the Lagrangian one must discover that terms without derivatives are not allowed (by symmetry), and moreover this provides an organizing principle, a counting scheme, in the infinite number of terms of the nonrenormalizable Lagrangian.

We now consider the case of QCD and briefly describe the construction of the effective Lagrangian in the chiral limit. If we consider N_f massless flavours (and disregard heavy quarks) the Lagrangian of QCD is

$$\mathcal{L}^{(0)}_{\rm QCD} = \bar{q}_L i \not{D} q_L + \bar{q}_R i \not{D} q_R - \frac{1}{4} G^a_{\mu\nu} G^{a\mu\nu} \qquad\qquad q = \begin{pmatrix} q_1 \\ \vdots \\ q_{N_f} \end{pmatrix} \tag{4}$$

and is symmetric under a large global group of transformations:

$$\tilde{G} = SU(N_f)_L \times SU(N_f)_R \times U(1)_V \times U(1)_A \ . \tag{5}$$

The two $U(1)$ factors are not relevant for the present discussion for two opposite reasons: $U(1)_V$ remains conserved and the baryon number is the corresponding conserved quantity, while $U(1)_A$ is broken by quantum effects (is anomalous). The group G we have been referring to in the previous discussion is, in the case of QCD:

$$G = SU(N_f)_L \times SU(N_f)_R \ . \tag{6}$$

The vacuum, on the other hand, is only invariant under the vector transformations, those for which $g_L = g_R$,where $(g_L, g_R) \in SU(N_f)_L \times SU(N_f)_R$. QCD in the chiral limit of N_f flavours is therefore expected to have $N_f^2 - 1$ massless particles with the quantum numbers of the broken generators, the axial ones, *i.e.* pseudoscalar particles.

The construction of the effective Lagrangian proceeds as follows:

1. The pions are collected into a matrix–valued field U belonging to the coset space G/H. Since $SU(N_f)_L \times SU(N_f)_R/SU(N_f)_V$ is isomorph to $SU(N_f)$ the matrix U is a $SU(N_f)$ matrix. The fact that it is an element of the coset is seen in its transformation properties under $SU(N_f)_L \times SU(N_f)_R$:

$$U' = g_R U g_L^\dagger \ . \tag{7}$$

The pion fields can be identified with the coefficients of the generators of $SU(N_f)$, if we write U as an exponential

$$U = \exp[i\phi/F] \ , \qquad \phi = \sum_{a=1}^{N_f^2-1} \phi_a \lambda_a \ , \tag{8}$$

where λ_a are the generators (the Gell-Mann matrices for $N_f = 3$ or the Pauli matrices for $N_f = 2$), and F is a constant with dimensions of an energy, necessary if we want to have the pion fields canonically normalized.

2. The effective Lagrangian must contain only the fields which are active in the energy range of interest (in our case the pions) and must respect the symmetries of the system. In this case this means constructing a Lagrangian containing the U matrix which is invariant under (7). Renormalizability is not an issue, and we can therefore have as many derivatives as we want. The possible terms with up to two derivatives are

$$\mathcal{L}_{\text{eff}} = f_1(U) + f_2(U)\langle U^\dagger \Box U\rangle + f_3(U)\langle \partial_\mu U^\dagger \partial^\mu U\rangle + O(\partial^4) \ , \tag{9}$$

where $\langle A\rangle$ stands for the trace of the matrix A.

3. After having listed all possible terms compatible with the symmetries of our system, we can look for redundancies and simplify our Lagrangian. In this case we have two observations: the Lagrangian (9) can only be invariant under $SU(N_f)_L \times SU(N_f)_R$ if the functions f_i do not depend at all from U – given the transformation property (7), any matrix element of U can be transformed at will, and the f_i remain invariant only if they do not depend on any of them. The term f_1 is therefore just a constant which we can drop. Under the integral sign the term proportional to f_2 can be transformed into the third one by

partial integration. In conclusion we are left with one single term if we consider terms with up to two derivatives:

$$\mathcal{L}_{\rm eff} = \mathcal{L}_2 + \mathcal{L}_4 + \mathcal{L}_6 + \dots \ , \qquad \mathcal{L}_2 = \frac{F^2}{4}\langle \partial_\mu U^\dagger \partial^\mu U \rangle \ , \tag{10}$$

where the constant factor has been chosen such that the kinetic term for the pion fields has canonical form (the generators of $SU(N_f)$ are normalized such that $\langle \lambda_a \lambda_b \rangle = 2\delta_{ab}$):

$$\mathcal{L}_2 = \frac{1}{2}\partial_\mu \phi_a \partial^\mu \phi_a + O(\phi^4) \ . \tag{11}$$

The constant F drops out in the kinetic term but appears in all the interaction terms. The effective Lagrangian becomes predictive once we can fix this constant. In order to do this we can look at the Noether currents of the symmetry of the system. At low energy they are represented by

$$V_i^\mu = i\frac{F^2}{4}\langle \lambda_i [\partial^\mu U, U^+] \rangle \qquad A_i^\mu = i\frac{F^2}{4}\langle \lambda_i \{\partial^\mu U, U^+\} \ . \tag{12}$$

The matrix elements of the axial currents between the vacuum and the pion fields are given by

$$\langle 0 | A_i^\mu | \pi^k(p) \rangle = i p^\mu \delta_{ik} F \ . \tag{13}$$

The constant F is the pion decay constant in the chiral limit. If we want to work to leading-order accuracy we can stop here and start calculating.

The effective Lagrangian contains only one term at this level, but since this is written with the matrix U, which is an exponential of the pion fields, it contains local interaction terms among as many pion fields as one wants. The interaction vanishes if we consider static fields, as implied by the Goldstone theorem. Here we already see an advantage of the effective Lagrangian method: to show that the Goldstone theorem implies a vanishing interaction among pions of zero momentum requires quite some labour. If we instead use the effective Lagrangian, we immediately find out that symmetry forbids the interaction among static pions. This is true for any other consequence of symmetry: with the effective Lagrangian one can derive it without "thinking", but just doing the usual quantum field theory calculations.

2.2 Explicit symmetry breaking

In the real world there are no massless hadrons – the lightest hadrons, however, are pseudoscalars which do form an octet (triplet) under $SU(3)_V$ ($SU(2)_V$) transformations, and so have the right quantum numbers for being the Goldstone bosons of chiral symmetry breaking in QCD with $N_f = 3$ ($N_f = 2$). The presence of a small mass for the candidate Goldstone bosons can be traced back to the presence of a small explicit chiral symmetry breaking term in the QCD Lagrangian – the only possible such term (if we want to consider a renormalizable theory) is a quark mass term.

$$\mathcal{L}^{\rm QCD} = \mathcal{L}_0^{\rm QCD} - \bar{q}\mathcal{M}q \tag{14}$$

where $\mathcal{M} = \mathrm{diag}(m_u, m_d, m_s)$ is the quark mass matrix. As long as this symmetry breaking term is small, we can consider the limit in which the latter vanishes and make an expansion around it. This amounts to an expansion in the quark masses. Whatever quantity we consider, we will express it as its value in the chiral limit plus corrections which are suppressed by powers of the quark masses. These corrections are given by matrix elements of the symmetry breaking term among appropriate external states – calculating these is difficult, as any matrix element in QCD, but since the external states are the unperturbed ones (in the chiral limit), and the symmetry breaking term has well defined transformation properties under chiral transformations, we can at least keep track of symmetry constraints on the correction terms.

Let us consider the mass of a generic hadron h, as an example:

$$M_h^2 = M_{h,0}^2 + \langle h|\bar{q}\mathcal{M}q|h\rangle + O(m_q^2) \ . \tag{15}$$

The corrections to the chiral limit value of the mass are determined by the matrix elements $\langle h|\bar{q}q|h\rangle$. For example, if we consider the pions, these are massless in the chiral limit, $M_{\pi,0}^2 = 0$ and the expansion reads [3]:

$$M_\pi^2 = -(m_u + m_d)\frac{1}{F_\pi^2}\langle 0|\bar{q}q|0\rangle + O(m_q^2) \tag{16}$$

where we have used a symmetry relation among matrix elements of $\bar{q}q$, a chiral Ward identity:

$$\langle \pi|\bar{q}q|\pi\rangle = -\frac{1}{F_\pi^2}\langle 0|\bar{q}q|0\rangle =: B_0 \ . \tag{17}$$

The quark condensate $\langle 0|\bar{q}q|0\rangle$ which determines the size of the pion mass squared to leading order in the quark mass expansion, is an order parameter for the spontaneous chiral symmetry breaking.

The effective Lagrangian method is a useful tool also in this circumstance, as it allows one to derive the symmetry constraints on the perturbative corrections in a systematic way. The implementation proceeds as follows: if one considers the quark mass matrix as an external, matrix-valued hermitian scalar field s, and assigns to it a specific transformation property under chiral transformations:

$$s' = g_R s g_L^\dagger \tag{18}$$

then one makes the QCD Lagrangian formally chiral invariant even in the presence of a mass term. Explicit symmetry breaking occurs because the "vacuum" around which one expands is not at $s = 0$ but rather at $s = \mathcal{M}$. The corresponding effective Lagrangian must also contain this external field s, it must still be chiral invariant, and explicit symmetry breaking is obtained by taking for s the constant value $s = \mathcal{M}$. The latter is considered as a perturbation and we expand in powers of s. The minimal number is one, and the only possible term in the effective Lagrangian is:

$$C\langle U^\dagger s + s^\dagger U\rangle = C\langle s(U^\dagger + U)\rangle \ , \tag{19}$$

where C is dimensionful constant. Considering directly the limit $s = \mathcal{M}$ we obtain the leading order effective Lagrangian:

$$\mathcal{L}_2 = \frac{F^2}{4}\left[\langle \partial_\mu U^+ \partial^\mu U\rangle + \langle 2B_0\mathcal{M}\left(U + U^+\right)\rangle\right] \ , \tag{20}$$

where we have fixed the constant $C = B_0F^2/2$ by calculating the value of the quark condensate in the effective theory.

With this effective Lagrangian we can make calculations and even compare these to measurements. For example we can evaluate the masses of the pseudo Goldstone bosons, and obtain:

$$\begin{aligned} M_\pi^2 &= (m_u + m_d)B_0 + O(m_q^2) \\ M_{K^+}^2 &= (m_u + m_s)B_0 + O(m_q^2) \\ M_{K^0}^2 &= (m_d + m_s)B_0 + O(m_q^2) \\ M_\eta^2 &= \frac{1}{3}(m_u + m_d + 4m_s)B_0 + O(m_q^2) \end{aligned} \tag{21}$$

These leading order expressions tell us a number of things. First of all they relate pseudoscalar mass ratios to quark mass ratios [4]:

$$\begin{aligned} M_K^2/M_\pi^2 &= (m_s+\hat{m})/2\hat{m} \qquad \Rightarrow m_s/\hat{m} = 25.9 \\ M_\eta^2/M_\pi^2 &= (2m_s+\hat{m})/3\hat{m} \qquad \Rightarrow m_s/\hat{m} = 24.3 \end{aligned} \tag{22}$$

and moreover they imply a prediction for the eta mass in terms of the masses of the pion and kaon:

$$3M_\eta^2 = 4M_K^2 - M_\pi^2 \ , \tag{23}$$

a relation first obtained by Gell-Mann and Okubo [5].

Besides masses, we can of course also calculate decay and scattering amplitudes. For example, the $\pi\pi$ scattering amplitude to leading order is given by [6]:

$$\begin{aligned} \langle \pi^c(p_3)\pi^d(p_4) \text{ out} \,|\, \text{in } \pi^a(p_1)\pi^b(p_2)\rangle &= \delta^{ab}\delta^{cd}A(s,t,u) + \delta^{ac}\delta^{bd}A(t,u,s) \\ &\quad + \delta^{ad}\delta^{bc}A(u,s,t) \ , \\ A(s,t,u) &= \frac{s-M_\pi^2}{F_\pi^2} \ , \end{aligned} \tag{24}$$

where the leading order expressions have been substituted with the physical pion mass and decay constant. Out of the isospin invariant amplitude $A(s,t,u)$ we can construct amplitudes of a given isospin in a given channel. Evaluating the amplitudes at threshold gives us (modulo a normalization factor) the (S wave) scattering lengths. The $I = 0$ and 2 scattering lengths are given by

$$a_0^0 = \frac{7M_\pi^2}{32\pi F_\pi^2} = 0.16 \ , \quad a_0^2 = -\frac{M_\pi^2}{16\pi F_\pi^2} = -0.045 \ . \tag{25}$$

Later we will see that these quantities are now known up to next-to-next-to-leading order, with very small uncertainties, and that there are a number of experiments running, which can test these precise predictions at a similar level of accuracy.

2.3 External fields

The introduction of a scalar field coupled to the scalar quark bilinear has served the purpose of dealing with explicit symmetry breaking. We have

also mentioned, in passing, that with the help of this scalar field we can evaluate in the effective theory matrix elements of $\bar{q}q$ – more precisely, we can derive automatically and systematically the symmetry relations respected by the matrix elements of this quark bilinear. For example, we are not able to calculate the quark condensate, but can relate the matrix element of $\bar{q}q$ between two one-pion states to the condensate. The same trick can be repeated with other operators, like the pseudoscalar bilinear $\bar{q}\gamma_5 q$, or the Noether currents $\bar{q}\lambda_a\gamma_\mu q$ and $\bar{q}\lambda_a\gamma_\mu\gamma_5 q$. This is the approach adopted by Gasser and Leutwyler [7, 8], who introduced the necessary external fields in QCD:

$$\mathcal{L} = \mathcal{L}_0^{\rm QCD} + \bar{q}\gamma^\mu(v_\mu + \gamma_5 a_\mu)q - \bar{q}(s - i\gamma_5 p)q \ , \tag{26}$$

then considered the generating functional, defined as the vacuum-to-vacuum amplitude with this action:

$$e^{{\rm i}Z[v,a,s,p]} \equiv \langle 0|Te^{{\rm i}\int d^4x\mathcal{L}}|0\rangle \ , \tag{27}$$

and stated that a low energy representation of the latter is conveniently obtained by evaluating the path integral over the pion fields with the action given by the effective Lagrangian constructed according to the principles we have discussed so far:

$$e^{{\rm i}Z[v,a,s,p]} = \mathcal{N}^{-1}\int [dU] e^{{\rm i}\int d^4x\mathcal{L}_{\rm eff}} \ . \tag{28}$$

The transformation property of the pseudoscalar external field is identical to that of the scalar one (18). Those of the external vector and axial-vector fields, could also be defined analogously. It is useful, however, to promote the chiral symmetry from global to local – while in this case the transformation properties of the scalar and pseudoscalar fields remain unchanged, those of the vector and axial-vector fields must be changed as follows

$$\begin{aligned} r_\mu = v_\mu + a_\mu \ &\stackrel{G}{\longrightarrow} \ r'_\mu = g_R r_\mu g_R^{-1} + i g_R \partial_\mu g_R^{-1} \ , \\ l_\mu = v_\mu - a_\mu \ &\stackrel{G}{\longrightarrow} \ l'_\mu = g_L l_\mu g_L^{-1} + i g_L \partial_\mu g_L^{-1} \ . \end{aligned} \tag{29}$$

The addition of the derivative term compensates the one generated by the kinetic term of the quark fields and makes QCD *locally* invariant under chiral transformations. Since the action itself is invariant, so is also the generating functional (not quite, because there is the chiral anomaly – we forget this for a moment and come back to it below):

$$Z[v',a',s',p'] = Z[v,a,s,p] \ . \tag{30}$$

The generating functional contains all possible Green functions of the quark bilinears to which the external fields are coupled. The invariance of the generating functional translates into relations among the Green functions – symmetry relations, *i.e.* Ward identities. The introduction of the derivative terms in the transformation of the vector and axial-vector fields has the consequence that the Ward identities contained in Eq. (30) also explicitly imply conservation of the axial and vector currents. In summary, Eq. (30) is a very compact way to write all possible Ward identities following from chiral invariance.

2.4 Chiral perturbation theory

With the effective Lagrangian method one aims at giving an explicit representation of the generating functional which respects the invariance property (30). This is obtained through a perturbative calculation of the path integral, evaluated with an effective Lagrangian containing only the low-energy degrees of freedom, *i.e.* the Goldstone bosons:

$$e^{iZ[v,a,s,p]} = \int [dU] e^{i\int dx \mathcal{L}_{\rm eff}(U,v,a,s,p)} \quad , \tag{31}$$

where the effective action $S_{\rm eff}[U,v,a,s,p] = \int dx \mathcal{L}_{\rm eff}(U,v,a,s,p)$ must contain all possible terms which are invariant under local chiral transformations – the latter is the only requirement we have to make if we want to be completely general. The effective Lagrangian which is invariant under local chiral transformations is obtained from Eq. (20) by transforming the normal derivatives into covariant ones and adding the pseudoscalar to the scalar external field – of course Eq. (20) represents only the leading order whereas the full effective Lagrangian contains terms with any number of derivatives and external fields:

$$\mathcal{L}_{\rm eff} = \mathcal{L}_2 + \mathcal{L}_4 + \mathcal{L}_6 + \ldots \quad \mathcal{L}_2 = \frac{F^2}{4}\langle D_\mu U^+ D^\mu U + \chi^\dagger U + U^\dagger \chi\rangle \ , \tag{32}$$

where $D_\mu U = \partial_\mu U - i r_\mu U + i U l_\mu$ and $\chi = 2B_0(s+ip)$.

The reasoning we just stated is absolutely general and for theories without anomalies this would be the end of the story. In QCD, however, the chiral symmetry is anomalous, and the generating functional is correspondingly noninvariant. It is possible, however, to evaluate exactly the noninvariant

term in the transformation – in this case the effective Lagrangian has to be constructed such that it reproduces the anomalous noninvariance of the generating functional. Wess and Zumino showed that in QCD this is indeed possible [9, 10, 11]. In the chiral counting the noninvariant term is first encountered at order p^4 – if we stop at order p^2 the Lagrangian in (32) is all there is.

The advantage of this formulation is that one directly deals with Green functions of quark bilinears, and these transform linearly under chiral transformations. Formulating symmetry properties working directly with the pions would be a lot more difficult, because these transform nonlinearly – moreover the exact transformation properties of the pions vary, depending on the representation one chooses for the pion fields. Indeed in this formulation of the effective Lagrangian method the pions move to the background and play the role only of integration variables in the path integral – in the foreground is the invariance property of the path integral. The latter is constructed in such a way that the Green functions it describes contain poles at the proper places – poles due to the exchange of pions. The residues of these poles give the matrix elements of pions. This is exactly what one would do if one were able to calculate directly the path integral of QCD (and it is exactly what one does with lattice calculations).

Another practical advantage of the external fields is that the W and Z bosons as well as the photons, couple to quarks exactly like the v and a fields we introduced. If we want to evaluate matrix elements involving quarks and these bosonic fields we just have to substitute the appropriate combinations of a and v with the W, Z, or photon fields and take the matrix element we are interested in.

3 Higher orders

In Eq. (32) we have specified only the leading order effective Lagrangian and just listed the next-to-leading and next-to-next-to leading term. If one wants to evaluate a physical quantity to a higher accuracy one needs to take these into account as well. But not only: as shown in Eq. (31), the generating functional (any Green function) of QCD at low energy is obtained by evaluating the path integral with the action specified by the effective Lagrangian – if we want to increase the accuracy of a calculation we have to evaluate also loop diagrams. The counting of loop diagrams in CHPT

has been discussed first by Weinberg [12] and can be summarized by the statement that every loop counts like order p^2: a complete calculation at order p^4 must necessarily include one-loop diagrams with vertices from the $\mathcal{L}_2$ Lagrangian as well as tree diagrams with the $\mathcal{L}_4$ Lagrangian. If we want to go up to order p^6 we must take into account two-loop diagrams with vertices only from $\mathcal{L}_2$ and one-loop diagrams with only one vertex from $\mathcal{L}_4$ and tree diagrams from $\mathcal{L}_6$.

These are the rules of the game – if one wants to play it one may stop reading here and start calculating. A student who has learned to do loop calculations within a renormalizable quantum field theory, can do it also using an effective Lagrangian. It is true, however, that the use of an effective Lagrangian for calculating loops poses some conceptual problems – a good student at this point will certainly ask himself: "when I calculate loops I integrate over all possible momenta, and if I use vertices and propagators from the effective Lagrangian, I am using these well beyond their range of validity. How can I trust such a calculation? What is its meaning?"

In this section I will provide answers to these questions by analyzing the loop calculation of one simple quantity, the scalar form factor of the pion, which is defined as

$$\langle \pi^i(p_1)\pi^j(p_2)|\hat{m}(\bar{u}u+\bar{d}d)|0\rangle =: \delta^{ij}\Gamma(t) \ , \quad t=(p_1+p_2)^2 \ , \tag{33}$$

where $\hat{m} = (m_s + m_d)/2$. This matrix element is relevant to the decay $h \to \pi\pi$, which would have been the main decay mode for a very light Higgs (of course this scenario is now experimentally excluded). The tree-level calculation of this matrix element is simple and leads to

$$\Gamma(t) = 2\hat{m}B = M_\pi^2 + \mathcal{O}(p^4) \ . \tag{34}$$

This result, which we worked out from the Lagrangian, is actually a consequence of a general theorem, due to Feynman and Hellman.[13] This states that the expectation value of the perturbation in an eigenstate of the total Hamiltonian determines the derivative of the energy level with respect to the strength of the perturbation:

$$\hat{m}\frac{\partial M_\pi^2}{\partial \hat{m}} = \langle \pi|\hat{m}\bar{q}q|\pi\rangle = \Gamma(0) \ . \tag{35}$$

The value of the form factor at zero momentum transfer is fixed by this theorem, and a simple power counting implies that at leading order the scalar

form factor is a constant – at order p^2 the theorem completely fixes the form factor. On the other hand no general principle forbids a dependence of the form factor on t (to the contrary, they imply it, as we will see), and to generate this we necessarily have to go beyond leading order.

Before starting the loop calculation, let us have a look at what happens to the form factor once we include tree-level contributions from higher-order terms in the Lagrangian. We now present the Lagrangian at order p^4 for the case in which one expands around $m_u = m_d = 0$ and keeping m_s at its physical value. In this case the chiral symmetry is $SU(2)_L \times SU(2)_R$. The Lagrangian of order p^2 remains unchanged – the only change is that the field U, and its logarithm ϕ are now 2×2 matrices. At order p^4 the Lagrangian is simpler, because we can use more trace identities to reduce the number of independent terms. For two light flavours this is a sum of seven terms[1]

$$\begin{aligned}\mathcal{L}_4 &= \ell_1 \frac{1}{4}\langle u_\mu u^\mu\rangle^2 + \ell_2 \frac{1}{4}\langle u_\mu u_\nu\rangle\langle u^\mu u^\nu\rangle + \ell_3 \frac{1}{16}\langle \chi_+\rangle^2 + \ell_4 \frac{i}{4}\langle u_\mu \chi_-^\mu\rangle \\ &- \ell_5 \frac{1}{2}\langle f_{-\mu\nu} f_-^{\mu\nu}\rangle + \ell_6 \frac{1}{4}\langle [u_\mu, u_\nu] f_+^{\mu\nu}\rangle - \ell_7 \frac{1}{16}\langle \chi_-\rangle^2 \,, \end{aligned} \tag{36}$$

where we have used the compact notation (with $U = u^2$):

$$\begin{aligned} u_\mu &= i u^\dagger D_\mu U u^\dagger = u_\mu^\dagger \\ \chi_\pm &= u^\dagger \chi u^\dagger \pm u \chi^\dagger u \\ \chi_\pm^\mu &= u^\dagger D^\mu \chi u^\dagger \pm u D^\mu \chi^\dagger u \\ f_\pm^{\mu\nu} &= u F_L^{\mu\nu} u^\dagger \pm u^\dagger F_R^{\mu\nu} u \ , \end{aligned} \tag{37}$$

and $F_{R,L}^{\mu\nu}$ are the field strengths of r_μ and l_μ . Only two of these seven terms contribute to the scalar form factor, those proportional to ℓ_3 and ℓ_4:

$$\Gamma_{[\ell_3,\ell_4]}(t) = \frac{M^2}{F^2}\left[-4M^2\ell_3 + t\ell_4\right] \ , \tag{38}$$

the calculation is recommended as an easy exercise, as is the calculation of the contribution of ℓ_3 to the pion mass. Once these two calculations are completed, one can then check that the Feynman–Hellman theorem is respected also in this case.

[1] If we disregard contact terms, i.e. those that depend only on the external fields. Notice also that in the theory with $N_f = 2$ and no electromagnetic interactions there is no chiral anomaly.

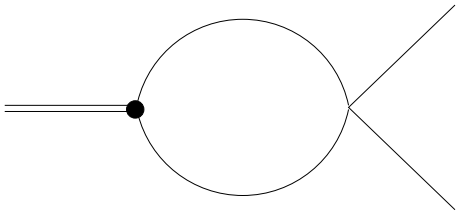

Figure 2: One-loop diagram for the scalar form factor of the pion. The double line stands for the scalar source, whereas single lines for pions.

The result in Eq. (38), a tree-level calculation with the next-to-leading order Lagrangian, is merely a statement about how the symmetry constrains this particular matrix element: at this order the scalar form factor can have at most a term linear in t. No symmetry relation exists between the constant term and the coefficient of the linear term, hence we have two different constants. The constant term is related to the derivative of the pion mass with respect to the strength of the symmetry-breaking term in the Lagrangian, whereas the coefficient of the linear term is related to the correction to the pion decay constant (again, it is a very good exercise to calculate the latter with the Lagrangian in Eq. (36)).

3.1 Loops and renormalization

Besides the less interesting tadpole graphs and the wave function renormalization there is only one graph for this process, the one shown in Fig. 2. Its structure is

$$\int \frac{d^4l}{(2\pi)^4} \frac{\{M^2, p^2, p\cdot l, l^2\}}{(l^2-M^2)((p-l)^2-M^2)} \;\Rightarrow\; xT(M^2) + yJ(t) \tag{39}$$

$$T(M^2) = \int \frac{d^4l}{i(2\pi)^4} \frac{1}{(M^2-l^2)} \;,\; J(t) = \int \frac{d^4l}{i(2\pi)^4} \frac{1}{(M^2-l^2)(M^2-(p-l)^2)} \;,$$

where $p = p_1 + p_2$. We have indicated, in the first integral, all the terms that can appear in the numerator, and, after the arrow, the two possible structures to which the various terms can be reduced. The momenta and masses in the numerator come from the four-pion vertex on the right-hand side of the diagram. The power counting for this integral shows that it represents a correction of order p^2 to the leading-order term (the integration

measure, which is of order p^4, is compensated by the two propagators) in agreement with the general counting rule stated by Weinberg [12].

If we expand the tadpole integral $T(M^2)$ and the loop integral $J(t)$ in a Taylor series in their respective arguments:

$$T(M^2) = a + bM^2 + \bar{T}(M^2) \ , \quad J(t) = J(0) + \bar{J}(t) \ , \tag{40}$$

only the first terms in the expansion are divergent, whereas both $\bar{T}(M^2)$ and $\bar{J}(t)$ are finite – this is easily seen by taking a sufficient number of derivatives on the loop integrals in Eq. (39). It is left as an exercise to the reader to show that the one-loop diagrams that we have neglected can only produce terms like $T(M^2)$. Therefore the contribution of the loop diagrams to the scalar form factor has the following structure:

$$\Gamma(t) \sim \frac{M^2}{F^2}\left[x_1 bM^2 + x_2 tJ(0) + x_1\bar{T}(M^2) + x_2\bar{J}(t)\right] \ . \tag{41}$$

The divergent part of the loop diagrams has exactly the same structure as the counterterm contribution calculated above: to remove it we simply need to define the counterterms properly (in this case the constants ℓ_3 and ℓ_4).

The principles we have followed in the construction of the effective Lagrangian only appealed to symmetry arguments, and therefore allowed for an infinite number of terms. Once this is accepted, there is no problem of principle in carrying through the renormalization program, because all counterterms necessary to remove the divergences must already be present: as anticipated, the difference between renormalizable and non-renormalizable theories is, in a sense, a technical detail.

To renormalize the form factor, the two counterterms ℓ_3 and ℓ_4 have to be defined in the following way

$$\ell_3 = \ell_3^r(\mu) - \frac{1}{2}\lambda \ , \quad \ell_4 = \ell_4^r(\mu) + 2\lambda \ , \tag{42}$$

where λ is divergent and dependent on the regularization method (in dimensional regularization, for example, it is defined by $\lambda = (c\mu)^{d-4}/(d-4)$, with c an arbitrary constant that defines the regularization scheme).

If we look at the definition of the operators in front of ℓ_3 and ℓ_4 we see that (as usual) they contain an infinite number of pion fields. For example they both contain a term with one scalar source and 6 pion fields. Does it mean that if we had calculated that matrix element to one loop we would

have found the same divergent part for ℓ_3 and ℓ_4? Or, to put it differently, that once we calculate the divergent part of the scalar form factor then we know the divergent part of all other matrix elements with a higher number of pion fields?

The answer is yes. Chiral symmetry puts a strong constraint on the divergences: they have to be chiral-invariant terms. This conclusion is originally due to Weinberg [12] on the basis of a highly plausible, but still heuristic argument. It is now put on a solid basis by the work of Leutwyler [18]: he proved that to calculate hadronic Green functions with an effective Lagrangian, such that they respect the Ward identities implied by the chiral symmetry, one necessarily has to start from a chiral-invariant effective Lagrangian. The non-trivial part in this statement is that it takes into account also quantum effects: anomalous symmetries show that it is not always true that a symmetry that exists at the classical level survives the quantum corrections – or vice versa, that to have a symmetrical quantum theory one necessarily has to start from a symmetrical classical Lagrangian.

The theorem applies also to the divergent part of the quantum corrections: they have to be chiral-invariant. A general theorem of quantum field theory states that the divergent part of a loop graph is a polynomial in the external masses and momenta. These two theorems lead to the conclusion that the divergences, order by order, can be reabsorbed by the chiral-invariant counterterms.

3.2 Chiral logarithms

If we expand the form factor in a Taylor series in t, we can write it in the following form:

$$\Gamma(t) = \Gamma(0)\left[1 + \frac{1}{6}\langle r^2\rangle_S^\pi t + O(t^2)\right] \quad . \tag{43}$$

The coefficient of the linear term, properly normalized, is called the scalar radius of the pion. Its size is a measure of the spatial extension of the pion when probed with a scalar source. We have stated above that the coupling constant that appears in this quantity, ℓ_4, also determines the first correction of the pion decay constant around the chiral limit. There is another piece of information on the scalar radius, which we can already gather from the

simple sketch of the loop calculation given above:

$$\langle r^2\rangle_S^\pi \sim J(0) = \int \frac{d^4l}{i(2\pi)^4}\frac{1}{(l^2-M^2)^2} \sim \ln\frac{M^2}{\Lambda^2} \quad , \tag{44}$$

namely that the scalar radius contains an infrared divergence. In the chiral limit this quantity diverges. This divergence should not be removed and does not represent a problem because it has a physical meaning, in the sense that when the pion becomes massless the cloud of pions surrounding any hadron (and therefore also the pion itself) extends to an infinite range. A quantity that measures the spatial extension of this cloud should indeed become infinite in the chiral limit. Notice that the scalar form factor is finite and remains finite also in the chiral limit – it is only the first derivative in t, calculated at $t = 0$ that diverges when the pion mass goes to zero.

These infrared divergences are present everywhere in pion physics, and in many cases they are among the most important physical effects (less so in the case of kaons). Their relevance was first pointed out by Li and Pagels[14]. The effective Lagrangian method provides a systematic way to calculate these effects. It is also interesting to observe that the coefficients of these logs are given by renormalization group equations – despite several attempts, and a thorough understanding of the relation between these logs and the renormalization group equations for nonrenormalizable theories, however, a way to resum the chiral logs has not yet been found [12, 15].

As we have seen in the above example, the chiral logs arise from the infrared region in the loop integrals – precisely the region where we should fully trust the vertices of our effective Lagrangian. One may be less at ease with the ultraviolet region of the loop integrals: there one has no justification for the use of the effective Lagrangian. On the other hand, through the process of renormalization, that part of the integrals is completely removed and substituted with unknown constants, the counterterms. As is sometimes said, these parametrize our ignorance of the physics that lies above the range of applicability of the effective Lagrangian.

3.3 Loops and unitarity

Until now the only part of the loop integrals that we have not analysed is the finite, analytically non-trivial part of the loop integral, the function $\bar{J}(t)$. Its presence is a consequence of unitarity, as we are now going to explain. According to the Watson theorem [16], above threshold but below

the inelasticity threshold, the phase of the scalar form factor is equal to the S-wave $\pi\pi$ phase shift with isospin $I = 0$, δ_0^0. As is well known, this theorem is a consequence of unitarity:

$$\mathrm{Im}\bar{\Gamma}(t) = \sigma(t)\bar{\Gamma}(t)t_0^{0*}(t) = \bar{\Gamma}(t)e^{-i\delta_0^0}\sin\delta_0^0 = |\bar{\Gamma}(t)|\sin\delta_0^0 \ , \tag{45}$$

where $\sigma(t) = [1 - 4M_\pi^2/t]^{1/2}$ and $\bar{\Gamma}(t) = \Gamma(t)/\Gamma(0)$, and t_0^0 is the $I = 0$ S-wave of $\pi\pi$ scattering. The unitarity relation (45) shows that the leading-order expression $\bar{\Gamma}(t) = 1$ cannot be the whole story: if we want an accurate description of the form factor away from $t = 0$, we need to include higher orders and, in particular, loops – imaginary parts can only be generated by loop graphs.

Notice that since at leading order $\bar{\Gamma}(t)$ is $\mathcal{O}(1)$, and the phase δ_0^0 is $\mathcal{O}(p^2)$, the $\mathcal{O}(p^2)$ imaginary part (which is a next-to-leading order correction) is completely fixed by leading-order quantities:

$$\mathrm{Im}\bar{\Gamma}^{(2)}(t) = \delta_0^{0(2)}(t) = \sigma(t)\frac{2t - M_\pi^2}{32\pi F_\pi^2} \ . \tag{46}$$

The use of the effective Lagrangian method to calculate the form factor guarantees that this relation is satisfied. The complete expression for the one-loop scalar form factor reads as follows:

$$\bar{\Gamma}(t) = 1 + \frac{t}{16\pi^2 F_\pi^2}(\bar{l}_4 - 1) + \frac{2t - M_\pi^2}{2F_\pi^2}\bar{J}(t) \ , \tag{47}$$

where $\bar{J}(t)$ is the subtracted one-loop integral (40). Its explicit expression reads:

$$\bar{J}(t) = \frac{1}{16\pi^2}\left[\sigma(t)\ln\frac{\sigma(t) - 1}{\sigma(t) + 1} + 2\right] \ . \tag{48}$$

The reader can now easily verify that the imaginary part of the form factor at this order indeed satisfies (46).

3.4 Dispersion relation for the scalar form factor

A real analytic function must be real on the real axis: the scalar form factor is non-analytic from threshold ($4M_\pi^2$) up to infinity. On the basis of very general arguments, which mainly use the causality principle, one can prove that as a function of t, the scalar form factor must be analytic everywhere else (see

Ref.[17] for a general discussion of the analyticity properties of amplitudes and Green functions). The non-analyticity of the form factor on the real axis can be further characterized, and described as a discontinuity:

$$\bar{\Gamma}(t+i\varepsilon) = \bar{\Gamma}^*(t-i\varepsilon) = |\bar{\Gamma}(t)|e^{i\delta_0^0(t)} \quad . \tag{49}$$

Given these analytic properties, we can write the following dispersion relation:

$$\bar{\Gamma}(t) = 1 + bt + \frac{t^2}{\pi}\int_{4M_\pi^2}^{\infty} \frac{dt'}{t'^2}\frac{\mathrm{Im}\bar{\Gamma}(t')}{t'-t} \quad , \tag{50}$$

where, for later convenience, we have subtracted the dispersive integral twice – we will come back to the issue of how many subtractions are necessary for the dispersive integral to converge. The dispersion relation shows that, if we know the subtraction constants (in this case only one, b) and the imaginary part on the real axis, we can reconstruct the scalar form factor everywhere on the complex plane.

It is no surprise that any perturbative calculation in a quantum field theory produces amplitudes and Green functions with the correct analytic properties. Using an effective field theory makes no difference: the form factor calculated to one loop in CHPT must have the correct analytic properties, and must satisfy (at the perturbative level) the dispersion relation (50). To convince ourselves that this is actually the case, let us first apply the chiral counting to the dispersion relation:

$$\bar{\Gamma}^{(0)} = 1 \ , \quad b \sim \mathcal{O}(1) \ , \quad \mathrm{Im}\bar{\Gamma}(t') = \mathcal{O}(p^2) \quad . \tag{51}$$

As we have seen the $\mathcal{O}(p^2)$ imaginary part is fully fixed by leading-order quantities, (46), and apart from an unconstrained polynomial term, the real part must be given by the dispersive integral over this known imaginary part. We leave it as an exercise to prove that this is true. For this it is useful to know the dispersive representation of the loop integral

$$\bar{J}(t) = \frac{t}{16\pi^2}\int_{4M_\pi^2}^{\infty}\frac{dt'}{t'}\frac{\sigma(t')}{t'-t} \quad . \tag{52}$$

In the previous section we showed that the renormalization procedure removes the contributions to the loop integrals where the momentum squared of the pions is large. This was reassuring because we cannot hope that our effective Lagrangian describes highly virtual pions well. In the present section

we are dealing with the contribution to the loop integrals from real pions: the dispersive integrals. As we have seen above, these extend all the way up to infinity, as required by analyticity. In the perturbative expansion that we are considering, the imaginary part of the form factor, which is in the integrand, is evaluated only to leading non-trivial order. This description of the imaginary part can be reasonably accurate only in the low energy region: still, in the integral, it is used all the way up to infinity. How can we trust the dispersive integral?

In fact, we do not. At least not for the contribution that comes from the high-energy region. Suppose we decide to remove the part of the dispersive integral from $\Lambda = 1$ GeV to infinity. We should then subtract from the form factor a term like:

$$\frac{t^2}{\pi}\int_{\Lambda^2}^{\infty}\frac{\delta_0^{0(2)}(t')}{t'^2(t'-t)} = \frac{t^2}{\pi}\int_{\Lambda^2}^{\infty}\frac{\delta_0^{0(2)}(t')}{t'^3}\left(1+\frac{t}{t'}+O(t^2)\right) \quad . \tag{53}$$

The Taylor expansion inside the integral can be safely performed because the CHPT calculation of the form factor is valid only for $t \ll \Lambda^2$. In the chiral language, this part of the dispersive integral can be represented as a polynomial series starting at order p^4, i.e. at an order which is beyond the accuracy at which we are presently working. This shows that worrying about these contributions to the dispersive integrals is futile: the only sensible way to improve the evaluation of the dispersive integral is to go one order higher in the chiral expansion. This would automatically give a representation of the form factor that contains the dispersive integral over the imaginary part correct up to order p^4. For those who are interested in the numerics, the first term in the Taylor expansion of the integral (53) is equal to:

$$\frac{t^2}{\pi}\int_{\Lambda^2}^{\infty}\frac{\delta_0^{0(2)}(t')}{t'^3} = 0.7\left[t(\,\mathrm{GeV}^2)^2\right] \quad , \tag{54}$$

which means, for $t = (0.5\,\mathrm{GeV})^2$ (which is about the upper limit of validity of the chiral expansion), a 4% correction to the leading-order result. Also numerically everything is well under control.

4 Applications

There are plenty of applications of the effective Lagrangian method and they are not restricted to particle physics, although the latter is the field where

the applications have reached the highest level of sophistication. If one wants to understand the phenomenology of strongly interacting particles at low energy, CHPT provides the proper framework to do it – not that there are no puzzles or experimental results which are not quite explained or understood, but even in these cases, the use of CHPT allows us to learn something from the puzzles. In this field the most interesting development of recent years is that experiments are reaching a high level of precision, and that the theory, stimulated by the perspective of having very precise data, has anticipated that and provided some remarkably precise predictions. This is obtained mainly in pion physics – the expansion in the mass of the strange quarks inevitably leads to a slowly converging series. In the following I will discuss two recent applications in pion physics which illustrate how, through a combined use of CHPT and dispersion relations one can reach a high level of precision, even for quantities which are in principle outside the region of application of CHPT, like the mass of resonances.

4.1 $\pi\pi$ scattering

In Sect. 2.2 we have discussed the leading order calculation of the $\pi\pi$ scattering amplitude and the numerical prediction (which is now 40 years old!) for the S-wave scattering lengths (25). In the meanwhile these have been calculated to NLO and NNLO. The formula of the S-wave scattering lengths to NLO is quite instructive [7]:

$$\begin{aligned} a_0^0 &= \frac{7M_\pi^2}{32\pi F_\pi^2}\left[1+\frac{M_\pi^2}{3}\langle r^2\rangle_S^\pi+\frac{200\pi F_\pi^2 M_\pi^2}{7}(a_2^0+2a_2^2)\right. \\ &\qquad \left. -\frac{M_\pi^2}{672\pi^2F_\pi^2}(15\bar{\ell}_3-353)\right] = 0.16\cdot 1.25 = 0.20 \\ 2a_0^0-5a_0^2 &= \frac{3M_\pi^2}{4\pi F_\pi^2}\left[1+\frac{M_\pi^2}{3}\langle r^2\rangle_S^\pi+\frac{41M_\pi^2}{192\pi^2F_\pi^2}\right] = 0.624 \ , \end{aligned} \tag{55}$$

as it shows explicitly that the core of the effective Lagrangian method is simply in establishing symmetry relations among different quantities. At leading order everything can be expressed in terms of the pion mass and decay constant – these fully specify the Lagrangian. At NLO we have seven new constants in the game, cf. Eq. (36), and they have a less direct physical meaning. Still, whenever they appear one can reexpress them in terms of other physical quantities and write a NLO chiral expression in a way which

makes the symmetry relations explicit. In Eq. (55) the two S-wave scattering lengths are expressed in terms of the pion scalar radius and the two D-wave scattering lengths ($a_2^{I=0,2}$) – plus the low-energy constant ℓ_3 which we have already encountered in Sec. 3, and which is the coefficient of the quadratic term in the quark mass expansion of the pion mass squared. A particular combination of the S-scattering lengths has a very simple relation to the scalar radius, as shown in Eq. (55). If one can pin down these quantities, one can work out a more accurate numerical prediction for the S-wave scattering lengths, as was originally done by Gasser and Leutwyler [7]. The remarkable outcome of the numerical estimate is that with respect to leading order, the NLO prediction for a_0^0 is 25% higher. For an expansion in $\hat{m}/1$ GeV $\sim 5\%$ this is quite a large correction. The reason for such a large correction is to

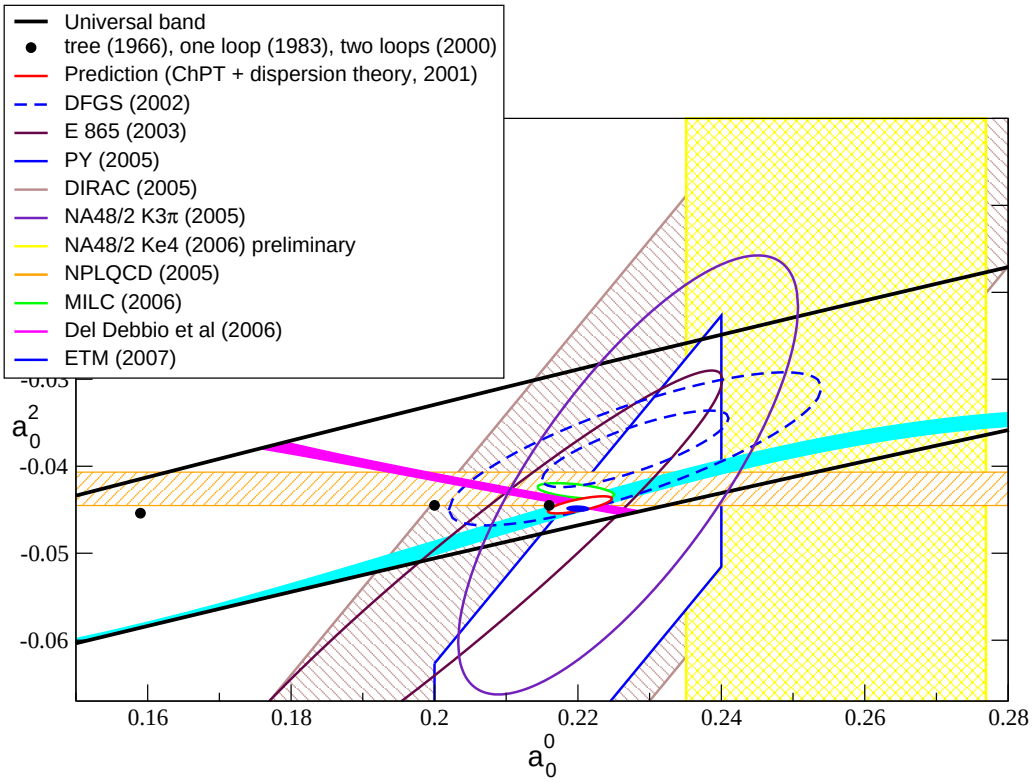

Figure 3: Theory predictions, lattice calculations and measurement of a_0^0 and a_0^2. See the text for more details.

be found in the presence of large chiral logs, also present in the scalar radius discussed in Sect. 3.1. The chiral logs in the scattering lengths are

$$a_0^0 = \frac{7M_\pi^2}{32\pi F_\pi^2}\left[1+\frac{9}{2}\ell_\chi+\ldots\right] \qquad a_0^2 = -\frac{M_\pi^2}{16\pi F_\pi^2}\left[1-\frac{3}{2}\ell_\chi+\ldots\right] \quad , \tag{56}$$

where

$$\ell_\chi = \frac{M_\pi^2}{16\pi^2 F_\pi^2}\ln\frac{\mu^2}{M_\pi^2} \quad , \tag{57}$$

and indeed the coefficient in a_0^0 happens to be quite large.

The $\pi\pi$ scattering amplitude has been later calculated even to NNLO [19], and an accurate estimate of the numerical prediction and the corresponding uncertainties has been worked out on the basis of a matching with dispersion relations [20]:

$$a_0^0 = 0.220 \pm 0.005, \qquad a_0^2 = -0.0444 \pm 0.0010 \quad . \tag{58}$$

These remarkably precise predictions are currently being tested by a number of experiments. A summary of the current situation is shown in Fig. 3 and is as follows

$$\begin{aligned}
a_0^0 &= 0.26 \pm 0.05 && [21]\\
a_0^0 &= 0.216 \pm 0.013 \pm 0.003 && [22]\\
|a_0^0 - a_0^2| &= 0.264\ {}^{+0.033}_{-0.020} && [23]\\
a_0^0 - a_0^2 &= 0.268 \pm 0.010 \pm 0.013 && [24]\\
a_0^0 &= 0.256 \pm 0.011 \qquad \text{— PRELIMINARY —} && [25] \quad .
\end{aligned} \tag{59}$$

After the old Geneva-Saclay experiment [21], the most recent ones have confirmed the CHPT prediction, although not yet reaching the same level of accuracy. The most recent number from the NA48/2 experiment [25], a measurement of the phase shift difference $\delta_0^0 - \delta_1^1$ in K_{e4} decays, and indirectly of the scattering length a_0^0, although preliminary, does show some tension with the prediction (58), and with the other K_{e4}-decay measurement of the E865 Collaboration from Brookhaven [22]. A completely different method to measure the difference $|a_0^0 - a_0^2|$ (through a measurement of the lifetime of pionium) is used by the DIRAC collaboration [23]. The same combination is measured by NA48/2 [24] in a cusp effect in $K \to 3\pi$ decays [26]. In Fig. 3 also the phenomenological analyses of Descotes-Genon *et al.* (entry DFGS

(2002) in the legenda, see Ref. [27]) and of Peláez and Ynduráin (entry PY (2005) in the legenda, see Ref. [28]) are shown.

In the same figure, besides the experimental numbers just discussed, also some lattice data have been plotted. One of them, from the NPLQCD collaboration [29] is a direct calculation on the lattice of the $I = 2$ S-wave scattering length and agrees well with the chiral prediction. The other three [30, 31, 32], are calculations of the constants $\ell_{3,4}$ which are two of the crucial inputs in the chiral calculation. The points on the plane corresponding to these calculations are obtained by using these numbers together with those of [20] for the remaining constants. All three agree with the chiral numbers. We stress, however, that the MILC numbers are the only ones obtained with three light sea (staggered) quark flavours – the other two simulations are done without strange quarks in the sea.

4.2 The σ meson

As we have discussed in Sec. 2.2, the lowest part of the QCD spectrum can be well understood within the effective Lagrangian framework: the lightest hadrons would be massless in the chiral limit and the pattern of their small masses is well explained by leading order formulae. It is of course an essential test of our understanding of strong interactions to be able to explain the whole hadronic spectrum, including resonances. One of the problems of this enterprise is to obtain reliable experimental information on these resonances. This appears to be particularly difficult for resonances with the quantum numbers of the vacuum: a look at the PDG [33] shows that, despite many relevant experimental results, the lightest resonance with isospin and spin zero – commonly known as the σ – is not known very well, cf. also Fig. 1. According to the PDG its mass lies between 400 and 1200 MeV and its half-width between 250 and 500 MeV. In this case, even if one were able to calculate the σ resonance parameters from first principles, it would hardly be a significant test to compare these to the "experimental" numbers. The main problem, in this case, is that in order to translate data into resonance parameters, one needs a theoretical description of the data – for most experimental situations it is difficult to have a model–independent parametrization, and as a result the same data can be interpreted in terms of wildly differing resonance parameters.

In a recent work [34] we have shown two things: first, that dispersion relations for scattering amplitudes do provide a model-independent way to

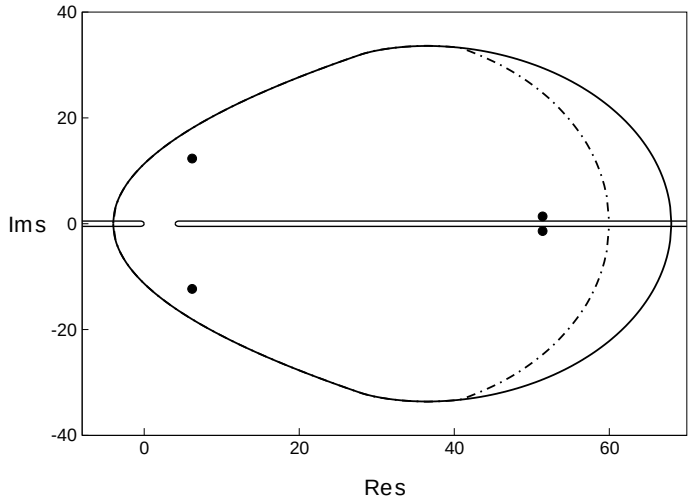

Figure 4: Domain of validity of the Roy equations for the partial waves $t_\ell^I(s)$ in the complex plane of the variable s, here expressed in pion mass squared units.

translate data into resonance parameters, and second, that in the case of the σ, the crucial input is represented by the two S-wave scattering lengths. The argument goes as follows: the Roy equations provide a dispersive representation for the partial waves of the $\pi\pi$ scattering amplitude which reads

$$\begin{aligned} t_0^0(s) &= a + (s-4M_\pi^2)\, b + \int_{4M_\pi^2}^{\Lambda^2} ds' \left\{ K_0(s,s')\, \mathrm{Im}\, t_0^0(s') \right. \\ &\left. + K_1(s,s')\, \mathrm{Im}\, t_1^1(s') + K_2(s,s')\, \mathrm{Im}\, t_0^2(s') \right\} + d_0^0(s)\,. \end{aligned} \tag{60}$$

This is a twice subtracted dispersion relation which takes into account crossing symmetry in order to

1. write the contribution of the left-hand cut in terms of the imaginary parts of other partial waves in the physical region;

2. write the two subtraction constants in terms of the two S-wave scattering lengths $a = a_0^0$, $b = (2\,a_0^0 - 5\,a_0^2)/(12M_\pi^2)$.

The so-called driving term $d_0^0(s)$ collects the dispersion integrals over the higher partial waves ($\ell \geq 2$), as well as the high energy end of the integral

over the S- and P-waves. The kernels $K_i(s,s')$ are known functions. The dispersive representation (60) is valid even for complex values of s, if these are inside the ellipse shown in Fig. 4, which is the region inside which the Roy equations have been rigorously proved to be valid. A direct evaluation of the representation (60) gives, however, the partial wave on the first Riemann sheet – the poles corresponding to resonances are to be found on the second sheet.

In order to reach the second sheet one can use the S matrix element $S_0^0(s) = 1 - 2\sqrt{4\,M_\pi^2/s - 1}\; t_0^0(s)$ and its unitarity property, which implies the relation

$$S_0^0(s)^{II} = 1/S_0^0(s)^I \quad . \tag{61}$$

This shows that the amplitude contains a pole on the second sheet if and only if $S_0^0(s)$ has a zero on the physical sheet. So, all we need to do is numerically evaluate Eq. (60) for complex values of s in the domain where it has been shown to hold and find out whether or not $S_0^0(s)$ has zeros there. The analysis presented in Ref. [34] yields the two pairs of zeros shown in Fig. 4. The one corresponding to the σ (the other is the $f_0(980)$) has parameters

$$M_\sigma = 441\,^{+16}_{-8}\ \text{MeV}\,, \quad \Gamma_\sigma = 544\,^{+18}_{-25}\ \text{MeV} \quad . \tag{62}$$

An analogous analysis has been recently applied to the κ [35] with similarly precise results.

We stress that the precision in the final numbers (62) is a direct consequence of the precise chiral prediction for the scattering lengths (58). The rest of the experimental input – the imaginary parts in the dispersive integrals in Eq. (60) – plays only a relatively minor role so that, despite relatively large experimental uncertainties, the corresponding contribution to the error in Eq. (62) is limited. One could, however, completely give up any use of theory in the calculation, and use only experimental input, even for the scattering lengths, cf. (59). This would increase the final error in (62) and shift somewhat the central values. The outcome, however, would be much more precise than what is still listed in the PDG.

5 Conclusions and outlook

In these lectures I have provided an introduction to effective field theories for systems with spontaneous symmetry breaking. I have mainly concentrated

on the foundations of the method and have emphasized the systematic and rigorous aspects of it: a proper application of effective field theories is strictly equivalent to a full exploitation of the symmetry properties of the underlying theory and of some general principles, like analyticity and unitarity, and *nothing else*. At the end I have discussed two recent applications, both related to $\pi\pi$ scattering. The two S-wave scattering lengths are two very interesting physical quantities which can be

1. predicted with very high accuracy within chiral perturbation theory (combined with Roy equations);

2. measured with very high accuracy and with different methods in running experiments;

3. calculated from first principles in lattice QCD.

Comparing the numbers obtained in these three manners represents a thorough test of our understanding of the strong interactions – the precision of this test is a rare exception in hadronic physics and illustrates well the power of the effective field theory method.

If these lectures have sparkled some interest in the reader, she will certainly want to read more. A thorough and detailed introduction to chiral perturbation theory can be found in Ref. [36], where the reader can also find an extensive bibliography and suggestions for further readings.

The very last word I want to spend to emphasize once more the universal character of effective field theories: applications cover all aspects of low–energy hadronic physics (in particular as far as the connections to lattice calculations and the related necessary extrapolations are concerned), but also extend to other very different physical systems, like antiferromagnets [37], or gravitational interactions [38].

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